Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2010 HSC ASSESSMENT TASK 3 (TRIAL HSC)

Mathematics Extension 2

Examiner: B. Weiss

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Barrett
- O Mr Trenwith
- O Mr Weiss

Student Number

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	/15	/15	/15	/15	/15	/15	/15	/15	/120	/100

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Question 1

(a) Evaluate

(i) $\displaystyle\int_0^1 \frac{4x+5\,dx}{(x+1)(2-x)}$ **3**

(ii) $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1}x\,dx$ **3**

(iii) $\displaystyle\int_0^a x\sqrt{a-x}\,dx$ **3**

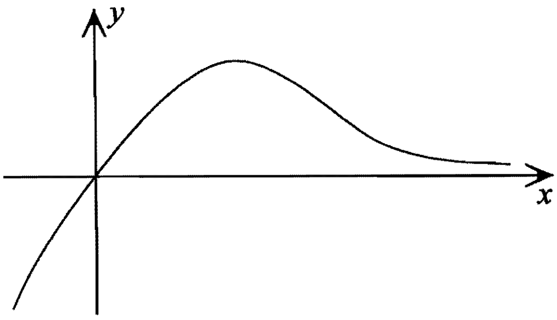
(b) (i) If $I_n = \displaystyle\int_0^1 x^n e^x\,dx$, show that **2**

$I_n = e - nI_{n-1}$, where n is a positive integer.

(ii) Hence evaluate $\displaystyle\int_0^{0\cdot 2} t^3 e^{5t}\,dt$. **4**

Question 2

(a)



The diagram shows the graph of $y = f(x)$, a curve which passes through the origin, and has a maximum turning point at $(1, 1)$.

Sketch on separate diagrams the graphs of

(i) $y = f(x) + 2$ **2**

(ii) $y = f(x + 2)$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = f(-x)$ **2**

(v) $y = |f(x)|$ **2**

(b) The graph of $f(x) = \dfrac{ax^2 + bx + c}{x^2 + qx + r}$ has the lines $x = 1$, $x = 3$ and $y = 2$ as asymptotes, and has a turning point at $(0, 1)$.

(i) Use this information to show that $f(x) = \dfrac{2x^2 - 4x + 3}{x^2 - 4x + 3}$. **2**

(ii) Sketch the graph of $y = f(x)$, showing clearly the coordinates of any points of intersection with the x-axis and the y-axis, the coordinates of any turning points, and the equations of any asymptotes. (There is no need to investigate points of inflexion). **3**

Question 3

(a) Find real numbers x, y such that $(2 - i)(3 + xi) = y + 5i$. **2**

(b) On an Argand Diagram, draw a neat sketch of the locus specified by: **3**

$$|z - i| = \sqrt{5}\,\mathrm{Re}(z)$$

(c) (i) Write down de Moivre's theorem for complex numbers. **1**

(ii) If $z_1 = \operatorname{cis}\theta_1$ and $z_2 = \operatorname{cis}\theta_2$, show that **3**

$$z_1 z_2^2 + \frac{1}{z_1 z_2^2} = 2\cos(\theta_1 + 2\theta_2).$$

(d) (i) Find the four solutions of $z^4 + 1 = 0$, writing them in the form $x + iy$. **4**

(ii) Hence, or otherwise, write $z^4 + 1$ as the product of two quadratic factors with real coefficients. **2**

Question 4

(a) (i) Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ intersect at right angles. **3**

(ii) Find the equation of the circle through the intersection of the two conics. **1**

(b) (i) Find the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $P(a\sec\varphi, b\tan\varphi)$. **3**

(ii) If this tangent passes through the focus of the ellipse **4**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b > 0)$$

show that it is parallel to one of the lines $y = x$ or $y = -x$, and that its point of contact with the hyperbola lies on the directrix of the ellipse.

(c) In a tidal river, the top of an old anchorage post measured 0·8 metres below the water level at high tide and 0·2 metres above the river level at low tide. High tide occurred at 6:30 am and low tide occurred at 12:35 pm on the day that the measurements were taken. The motion of the tide can be assumed to be Simple Harmonic. **4**
Between high tide and the next low tide on this day, when was there at least 0·5 metres of water above the top of the old anchorage post?
Express your times to the nearest minute.

Question 5

(a) A polynomial $P(x)$ has a double root at $x = \alpha$.

(i) Prove that $P'(x)$ also has a root at $x = \alpha$. **2**

(ii) The polynomial $Q(x) = x^4 - 6x^3 + ax^2 + bx + 36$ has a double root at $x = 3$. Find the values of a and b. **3**

(iii) Factorise $Q(x)$ over the complex field. **2**

(b) The polynomial $x^3 - 4x + 10$ has roots α, β and γ.

(i) Find the polynomial equation with roots α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

(c) The equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β and γ.
Find the equations with roots

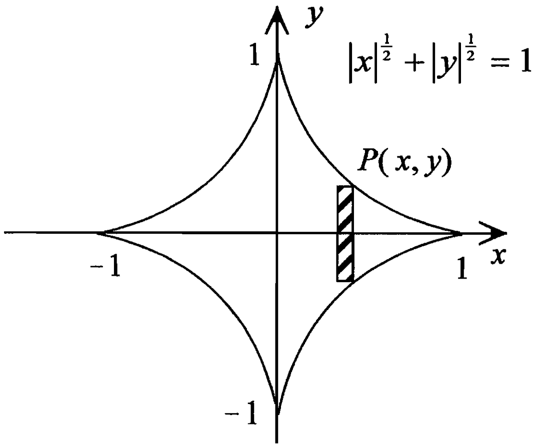
(i) $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$, $\alpha + \beta + 2\gamma$ **3**

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$. **2**

Question 6

(a) Use the method of cylindrical shells to find the volume of the solid generated by rotating the region bounded by $y = \ln x$, the x-axis and $1 \le x \le e$ about the y-axis. **4**

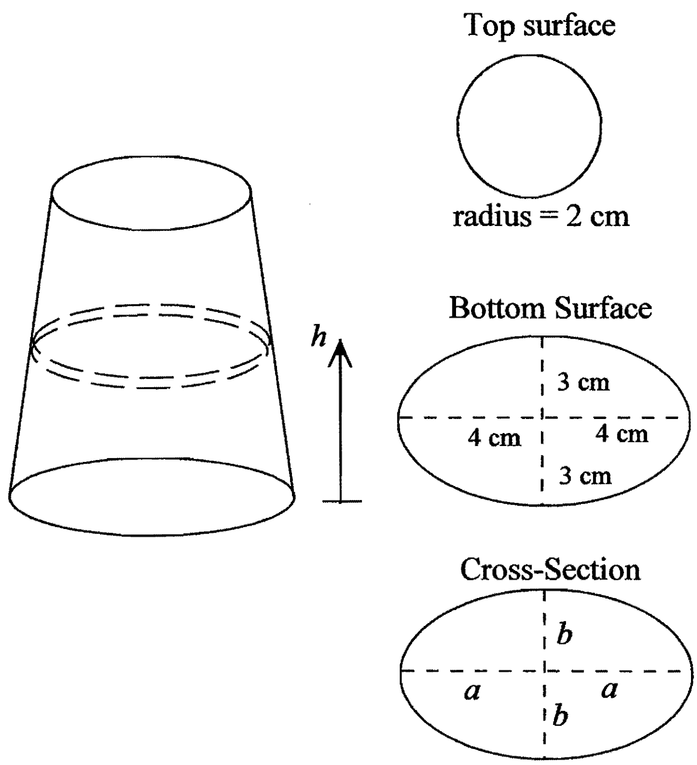
(b) The horizontal base of a solid is the region enclosed by the curve $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = 1$. Vertical cross-sections taken perpendicular to the x-axis are squares with one side in the base.



(i) Show that the volume of the solid is given by $V = 8\int_0^1 \left(1 - \sqrt{x}\right)^4 dx$. **2**

(ii) Use the substitution $u = 1 - \sqrt{x}$ to evaluate this integral. **3**

(c) A solid has an elliptical base and a circular top, as shown below. The height of the solid is 20 cm. All other dimensions are shown on the diagram.



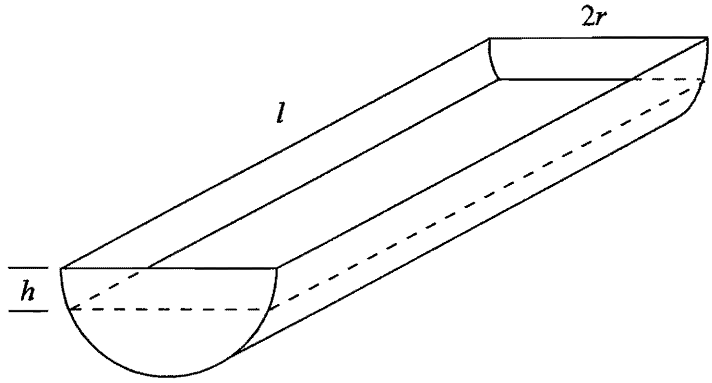
Each cross-section parallel to the base is an ellipse. A slice is taken h cm from the base.

(i) Find an expression for the area of each slice in terms of h. **4**
(You can assume that the area of an ellipse with semi-major and semi-minor axes of a and b respectively is πab.)

(ii) Hence find the exact volume of the solid. **2**

Question 7

(a) A water trough takes the shape of a hollow semi-circular prism with length l and radius r. It is placed on horizontal ground and filled with water. The surface of the water is at a distance h below the top of the trough, as shown in the diagram.



(i) Show that the area A of the flat surface of water is given by **2**

$$A = 2l\sqrt{r^2 - h^2}.$$

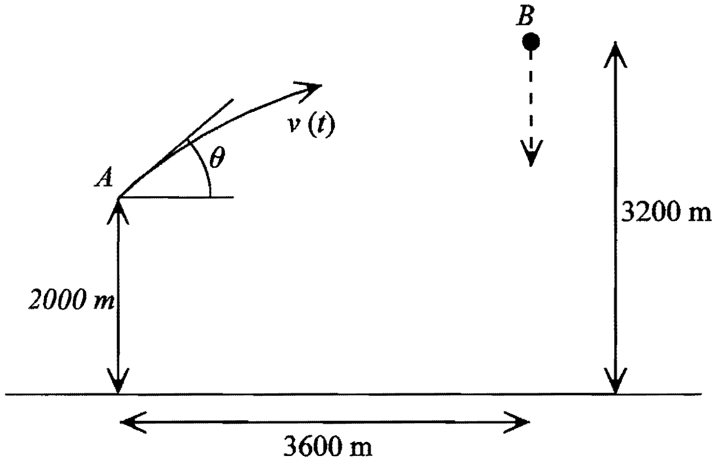
(ii) Show that the volume V of water in the trough is given by **2**

$$V = l\left(r^2 \cos^{-1}\left(\frac{h}{r}\right) - h\sqrt{r^2 - h^2}\right).$$

(iii) If the water level is falling, show that $\frac{dV}{dt} = -2l\sqrt{r^2 - h^2}\frac{dh}{dt} = -A\frac{dh}{dt}$. **3**

(iv) On a sunny day, the rate of evaporation at any time $\left(\text{and hence } -\frac{dV}{dt}\right)$ is proportional to A. Show that the water level falls at a constant rate. **1**

(b) An aeroplane A flying at a height of 2000 metres observes a stationary blimp B at a height of 3200 metres. Simultaneously, an object is dropped from the blimp, and the plane fires a projectile towards it at a speed of 240 m/s and at an angle of θ to the horizontal. The horizontal distance between the plane and the blimp is 3600 metres at the time that the projectile is fired.



The origin of coordinates O is taken to be the point on the ground below A. The particle's coordinates at time t (secs) are given by:

$$x = 240\,t\cos\theta$$

$$y = 2000 + 240\,t\sin\theta - \frac{gt^2}{2}.$$

The coordinates of the dropped object at time t are:

$$x = 3600$$

$$y = 3200 - \frac{gt^2}{2}$$

(You may use $g = 10$ m/s^2)

(i) What is the angle θ at which the projectile must be fired to intercept the object, and how long does it take to reach it? **3**

(ii) At what height does the projectile intercept the object? **1**

(c) Using mathematical induction, prove that **3**

$$\sum_{r=1}^{n} r^3 < n^2(n+1)^2$$

Question 8

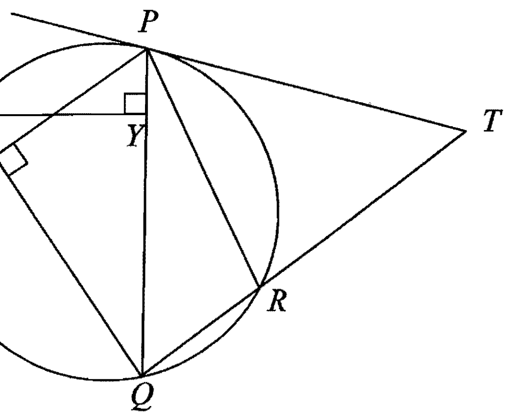
(a) A man notices two towers, one due north, and one on a bearing of $\theta°$, where $0 \le \theta \le 90°$. The angle of elevation β of both towers is the same, but the height of one tower is twice the height of the other. Show that **4**

$$\cos\theta = \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}$$

where α is the angle of elevation of the top of the taller tower from the top of the shorter.

(b) In the diagram below, TP is a tangent to the circle at P, and TQ is a secant cutting the circle at R.

SQ is a chord of the circle such that PX and SY are perpendicular to SQ and PQ respectively.

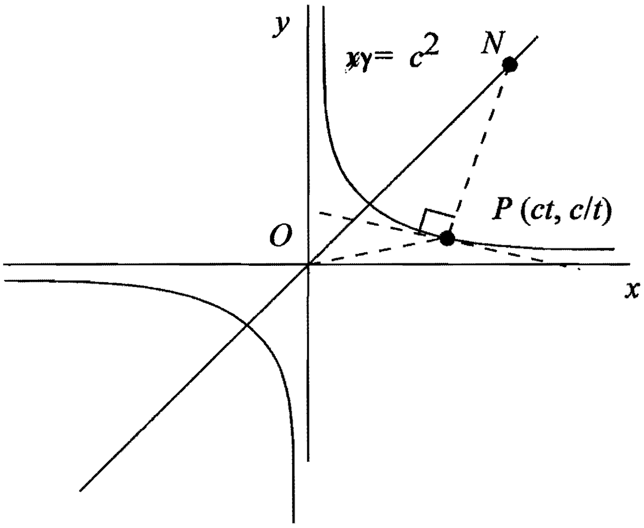


(i) Prove that $\angle TRP = \angle TPQ$. **3**

(ii) Explain why $SPYX$ is a cyclic quadrilateral, and give the diameter of the circle. **1**

(iii) Prove that $\angle PYX = \angle PRQ$. **3**

(c) The diagram below shows the hyperbola $xy = c^2$. The point $P\left(ct, \frac{c}{t}\right)$ lies on the curve, where $t \neq 0$. The normal at P intersects the straight line $y = x$ at N. O is the origin.



(i) Prove that the equation of the normal is $y = t^2 x + \frac{c}{t} - c^3$. **2**

(ii) Find the coordinates of N. **1**

(iii) Show that triangle OPN is isosceles. **2**

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Q1) a) i/ $\int_0^1 \frac{4x+5}{(x+1)(2-x)}\,dx$

Now $\frac{4x+5}{(x+1)(2-x)} = \frac{a}{x+1} + \frac{b}{2-x}$

when $x = 2$ $\quad 13 = 3b$

$b = \frac{13}{3}$

when $x = -1$ $\quad 1 = 3a$

$a = \frac{1}{3}$ $\quad$ 1

$\therefore \int_0^1 \frac{4x+5}{(x+1)(2-x)}\,dx = \frac{1}{3}\int_0^1 \left(\frac{1}{x+1} + \frac{13}{2-x}\right)dx$

$= \frac{1}{3}\Big[\ln(x+1) - 13\ln(2-x)\Big]_0^1$ $\quad$ 1

$= \frac{1}{3}\ln 2 + \frac{13}{3}\ln 2$

$= \frac{14}{3}\ln 2$ $\quad$ 1

ii/ $\int_0^{1/2} \sin^{-1}x\,dx$

$= x\sin^{-1}x\Big]_0^{1/2} - \int_0^{1/2} x.\frac{1}{\sqrt{1-x^2}}\,dx$ $\quad$ 1

$= \frac{1}{2}\times\frac{\pi}{6} - \frac{1}{2}\left[\frac{(1-x^2)^{1/2}}{-1/2}\right]_0^{1/2}$ $\quad$ 1

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ $\quad$ 1

iii/ $\int_0^a x\sqrt{a-x}\,dx$

Let $u = a - x$

$du = -dx$ $\quad$ 1

$\therefore -\int_a^0 (-u+a)\,u^{1/2}\,du$

$= \int_0^a -u^{3/2} + au^{1/2}\,du$ $\quad$ 1

$= -\frac{2}{5}u^{5/2} + \frac{2au^{3/2}}{3}\Big]_0^a$

$= -\frac{2}{5}a^{5/2} + \frac{2}{3}a^{5/2}$

$= \frac{4a^{5/2}}{15}$ $\quad$ 1

b) i/ $I_n = \int_0^1 x^n e^x\,dx$ $\quad$ 1

$= x^n e^x\Big]_0^1 - n\int_0^1 e^x x^{n-1}\,dx$

$= e - n\int_0^1 e^x x^{n-1}\,dx$

$\therefore I_n = e - n\,I_{n-1}$ $\quad$ 1

Q1) b) ii

$$I_n = e - n I_{n-1}$$

Let $u = 5t$

$du = 5\,dt$

$$\int_0^{1/5} t^3 e^{5t}\,dt$$

$$\therefore \frac{1}{5}\int_0^1 \left(\frac{u}{5}\right)^3 \times e^u \,du$$

$$= \left.\frac{t^n e^{5t}}{5}\right]_0^{1/5} - \frac{n}{5}\int_0^{1/5} t^{n-1} e^{5t}\,dt$$

$$= \left(\frac{1}{5}\right)^n \frac{e}{5} - \frac{n}{5} I_{n-1}$$

$$I_3 = \frac{e}{625} - \frac{3}{5} I_2$$

$$I_2 = \frac{e}{125} - \frac{2}{5} I_1$$

$$I_1 = \int_0^{1/5} t e^{5t}\,dt$$

$$= \left.\frac{t}{5} e^{5t}\right]_0^{1/5} - \int_0^{1/5} \frac{e^{5t}}{5}\,dt$$

$$= \frac{1}{25} e - \left[\frac{e^{5t}}{25}\right]_0^{1/5}$$

$$= \frac{e}{25} - \frac{e}{25} + \frac{1}{25}$$

$$= \frac{1}{25}$$

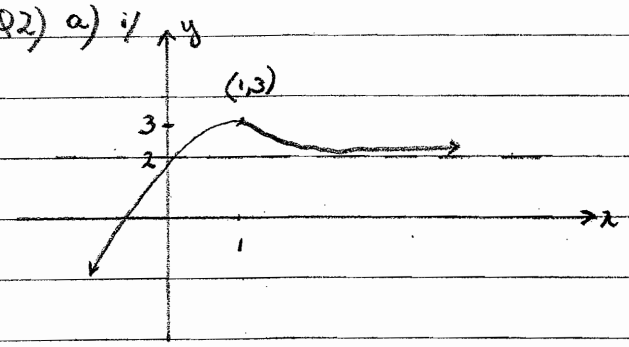
$$I_2 = \frac{e}{125} - \frac{2}{5} \times \frac{1}{25}$$

$$= \frac{e}{125} - \frac{2}{125}$$

$$I_3 = \frac{e}{625} - \frac{3}{5}\left(\frac{e}{125} - \frac{2}{125}\right)$$

$$= \frac{6}{625} - \frac{2e}{625}$$

Q2) a) i/



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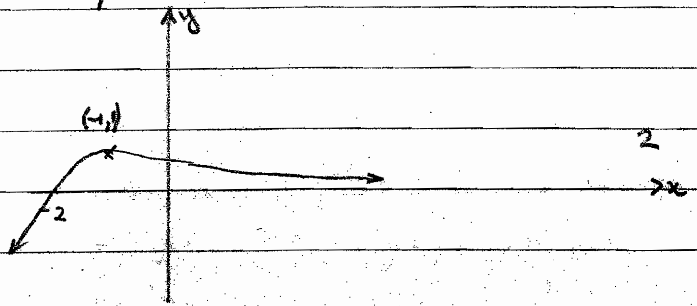
ii/



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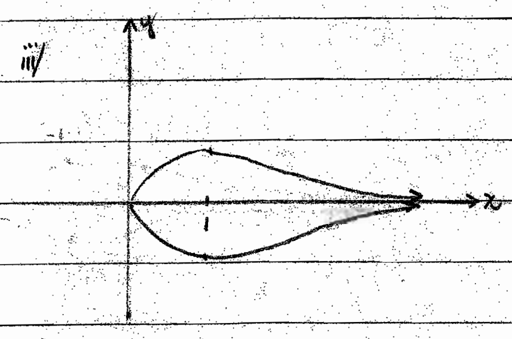
iii/



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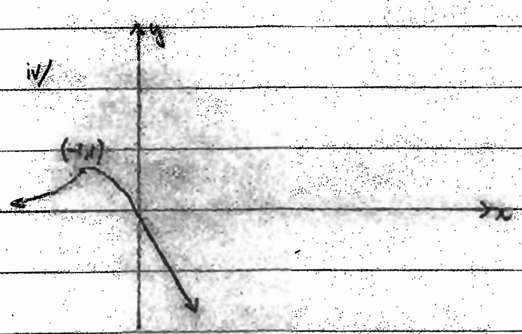
iv/



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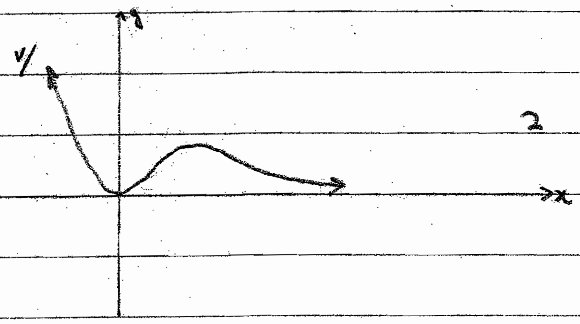
v/



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b) i/ $f(x) = \dfrac{ax^2+bx+c}{x^2+qx+r}$

as $x = 1, 3$ are asymptotes

$\therefore\ q = -4,\ r = 3$ 1

as $y = 2$ is an asymptote

$\therefore\ a = 2$

$f(0) = 1 \quad \therefore\ c = 3$ 1

$f'(0) = 0 \quad \therefore\ b = -4$

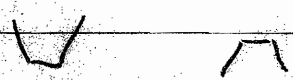
$\therefore\ f(x) = \dfrac{2x^2-4x+3}{x^2-4x+3}$

ii/ $f'(x) = \dfrac{-2x(2x-3)}{(x^2-4x+3)^2}$

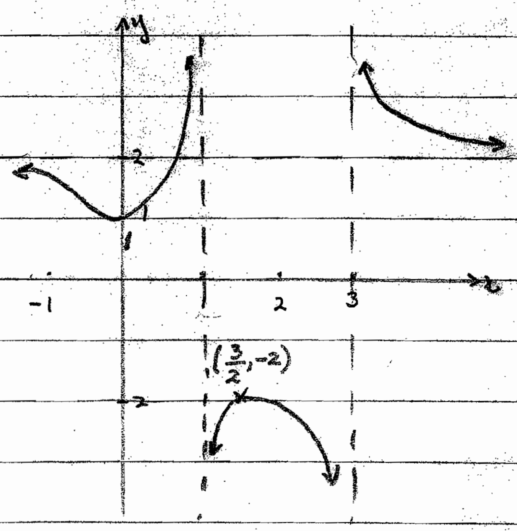
$\therefore\ f'(x) = 0$ when $x = 0, \frac{3}{2}$

x	-1	0	1	$\frac{3}{2}$	2
$f'(x)$	<0	$=0$	>0	$=0$	<0

1



$\therefore$ min $(0,1)$ max $(\frac{3}{2}, -2)$



1 for asymptotes

Q3) a) $(2-i)(3+xi) = y+5i$

$6 + 2xi - 3i + x = y+5i$

$(6+x) + i(2x-3) = y+5i$ 1

$\therefore\ 6+x = y$

$2x-3 = 5$

$x = 4$

$\therefore\ y = 10$ 1

b) $|z-i| = \sqrt{5}\ R(z)$

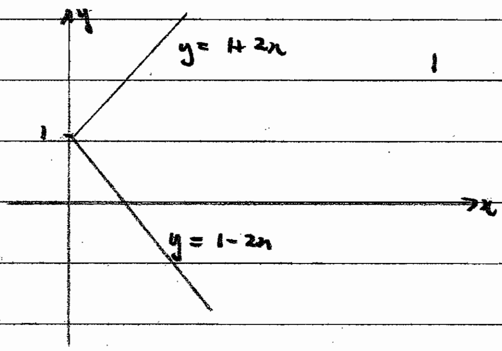
$x^2+(y-1)^2 = 5x^2$

$4x^2 - (y-1)^2 = 0$

$(y-1)^2 = 4x^2$

$y-1 = \pm 2x$ 1

$\therefore\ y = 1 \pm 2x$



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Since $|z-i| \geq 0$ 1

$x \geq 0$

c) i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ 1

ii) $L.H.S. = z_1 z_2^2 + \frac{1}{z_1 z_2^2}$

$= cis\,\theta_1 (cis\,\theta_2)^2 + \frac{1}{cis\,\theta_1 (cis\,\theta_2)^2}$

$= cis\,\theta_1\ cis\,2\theta_2 + cis(-\theta_1)\ cis(-2\theta_2)$ 1

$= cis(\theta_1+2\theta_2) + cis(-\theta_1-2\theta_2)$

$= \cos(\theta_1+2\theta_2) + i\sin(+\theta_1+2\theta_2)$

$+ \cos(-\theta_1-2\theta_2) + i\sin(-\theta_1-2\theta_2)$ 1

$= \cos(\theta_1+2\theta_2) + i\sin(\theta_1+2\theta_2)$

$+ \cos(\theta_1+2\theta_2) - i\sin(\theta_1+2\theta_2)$

$= 2\cos(\theta_1+2\theta_2)$ 1

$= RHS$

#

d) i) $z^4 = -1$

if $z = r\,cis\,\theta$

$(r\,cis\,\theta)^4 = -1$

$r^4\,cis\,4\theta = -1$

$|z^4| = |z|^4 = -1$

$\Rightarrow z = r = 1$ 1

$\therefore\ cis\,4\theta = -1$

$\therefore\ \cos 4\theta = 1$

$4\theta = \pi + 2k\pi$

$\theta = \frac{\pi}{4} + \frac{k\pi}{2}$ 1

$\therefore\ z_1 = cis\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$

$z_2 = cis\frac{3\pi}{4} = \frac{-1+i}{\sqrt{2}}$ 1

$z_3 = cis-\frac{\pi}{4} = \frac{1-i}{\sqrt{2}}$ 1

$z_4 = cis\left(-\frac{3\pi}{4}\right) = \frac{-1-i}{\sqrt{2}}$

ii) $z^4+1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$

$= (z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)$

$= (z^2 - z(z_1+\bar{z}_1) + z_1\bar{z}_1)$ 1

$(z^2 - z(z_2+\bar{z}_2) + z_2\bar{z}_2)$

$= (z^2 - z(\frac{2}{\sqrt{2}})+1)(z^2 - z(\frac{-2}{\sqrt{2}})+1)$

$= (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$

1

Q4) a) i) $4x^2+9y^2=36$ ①

$8x+18y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-8x}{18y}=-\frac{4x}{9y}$ 1

$4x^2-y^2=4$ ②

$8x-2y\frac{dy}{dx}=0$

$\therefore \frac{dy}{dx}=\frac{8x}{2y}=\frac{4x}{y}$ 1

① − ② $10y^2=32$

$y_1^2=\frac{16}{5} \quad \therefore x_1^2=\frac{9}{5}$ 1

at pt of intersection

$m_1\times m_2=\frac{-4x}{9y}\times\frac{4x}{y}=\frac{-16x^2}{9y^2}$

$=\frac{-16\times\frac{9}{5}}{9\times\frac{16}{5}}$ 1

$=-1$

∴ ⊥'ar.

ii/ centre (0,0) radius $x_1^2+y_1^2=\frac{9}{5}+\frac{16}{5}=5$ 1

∴ eqn of circle $x^2+y^2=5$

b) i/ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\phi, b\tan\phi)$

$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{2x}{a^2}\times\frac{b^2}{2y}$

$=\frac{b^2a\sec\phi}{a^2b\tan\phi}$

$=\frac{b\sec\phi}{a\tan\phi}$ 1

$y-b\tan\phi=\frac{b\sec\phi}{a\tan\phi}(x-a\sec\phi)$

$a\tan\phi\, y-ab\tan^2\phi=b\sec\phi\, x-ab\sec^2\phi$ 1

$a\tan\phi\, y-b\sec\phi\, x=ab(\tan^2\phi-\sec^2\phi)$

$\therefore a\tan\phi\, y-b\sec\phi\, x=-ab$ 1

ii) if it passes thru focus of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

" " " $(ea,0)$

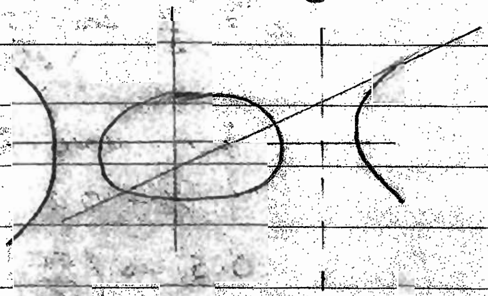
$b\sec\phi\, ae-0=ab$

$\sec\phi=\frac{1}{e}$ 1

$\therefore \tan\phi=\sqrt{1-e^2}$

∴ P$(a\sec\phi, b\tan\phi)$

$x=a\times\frac{1}{e}=\frac{a}{e}$ ∴ on directrix



$m=\frac{b\tan\phi-0}{\frac{a}{e}-ae}$ 1

$=\frac{eb\tan\phi}{a-ae^2}$

$=\frac{be\sqrt{\sec^2\phi-1}}{a(1-e^2)}$

$=\frac{b(1-e^2)^{-1/2}}{a}$ 1

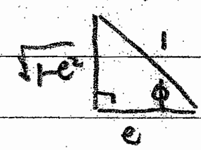
where $b=a\sqrt{1-e^2}$

$=\frac{(a\sqrt{1-e^2})(1-e^2)^{-1/2}}{a}$

$=1$ 1

∴ parallel to $y=x$

OR $\sec\phi=\frac{1}{e}$



$\tan\phi=\sqrt{1-e^2}$

$m=\frac{b\sec\phi}{a\tan\phi}=\frac{\frac{b}{e}}{\frac{a\sqrt{1-e^2}}{e}}=\frac{b}{a\sqrt{1-e^2}}=\pm1$

Q5) a) i) $P(x) = (x-\alpha)^2 Q(x)$

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$ 1

$= (x-\alpha)(2Q(x) + (x-\alpha)Q'(x))$

$\therefore P'(x)$ has a factor $x-\alpha$

$\therefore x = \alpha$ is a root 1

ii) $Q(x) = x^4 - 6x^3 + ax^2 + bx + 36$

$Q'(x) = 4x^3 - 18x^2 + 2ax + b$

Now $Q(3) = 0$ and $Q'(3) = 0$ 1

$Q(3) = 81 - 162 + 9a + 3b + 36 = 0$

$9a + 3b = 45$

$Q'(3) = 108 - 162 + 6a + b = 0$

$6a + b = 54$ (2) 1

$3a + b = 15$ (1)

$\therefore 3a = 39$

$a = 13$

$b = -24$ 1

iii) $x^4 - 6x^3 + 13x^2 - 24x + 36 = (x-3)^2(ax^2+bx+c)$

$a = 1$

$9c = 36 \quad c = 4$ 1

$9b - 6c = 24$

$b = 0$

$\therefore Q(x) = (x-3)^2(x^2+4)$

$= (x-3)^2(x+2i)(x-2i)$ 1

b) i/ $x^3 - 4x + 10$

$x = \alpha^2, \beta^2, \gamma^2$

$\sqrt{x} = \alpha, \beta, \gamma$ 1

$(\sqrt{x})^3 - 4\sqrt{x} + 10 = 0$

$x\sqrt{x} - 4\sqrt{x} = -10$

$\sqrt{x}(x-4) = -10$ 1

$x(x-4)^2 = 100 \Rightarrow x^3 - 8x^2 + 16x - 100 = 0$

$\therefore x^3 - 8x^2 + 16x - 100 = 0$

ii/ $\alpha^2 + \beta^2 + \gamma^2 = 8$ 1

c) i/ $x^3 - 3x^2 - x + 2 = 0$

$\therefore \alpha + \beta + \gamma = 3$ 1

$x = 2\alpha + \beta + \gamma = 3 + \alpha$

$\therefore \alpha = (x-3)$ is a root 1

$\therefore (x-3)^3 - 3(x-3)^2 - (x-3) + 2 = 0$

$x^3 - 12x^2 + 44x - 49 = 0$ 1

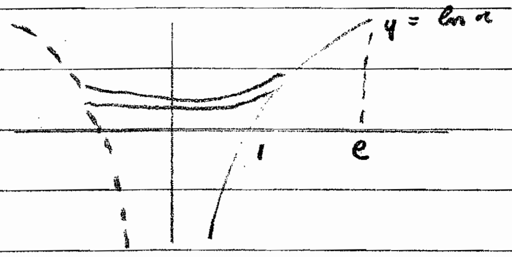
ii/ $x = \frac{1}{\alpha}$ satisfies

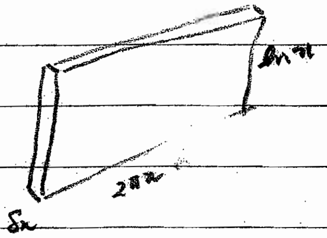
$\alpha = \frac{1}{x}$ 1

$\left(\frac{1}{x}\right)^3 - 3\left(\frac{1}{x}\right)^2 - \frac{1}{x} + 2 = 0$

$\therefore 1 - 3x - x^2 + 2x^3 = 0$ 1

Q6) a)





1

$V = \lim_{\delta x \to 0} \sum_{1}^{e} 2\pi x \ln x \, \delta x$

$= 2\pi \int_1^e x \ln x \, dx$ 1

$= 2\pi \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{1}{x} \, dx$

$= 2\pi \left[\frac{e^2}{2} \right] - 2\pi \left[\frac{x^2}{4} \right]_1^e$ 1

$= \pi e^2 - \frac{\pi e^2}{2} + \frac{\pi}{2}$

$= \frac{\pi}{2}(e^2+1)\ u^3$ 1

b) i) Area of cross-section $(2y)^2$

$= 4\left(1 - |x|^{1/2}\right)^4$ 1

$V = \lim_{\delta x \to 0} \sum_{-1}^{1} 4\left(1-|x|^{1/2}\right)^4 \delta x$

$= 8 \int_0^1 (1-\sqrt{x})^4 \, dx$ 1

ii) Let $u = 1 - \sqrt{x}$

$x = (1-u)^2$

$dx = -2(1-u)\,du$

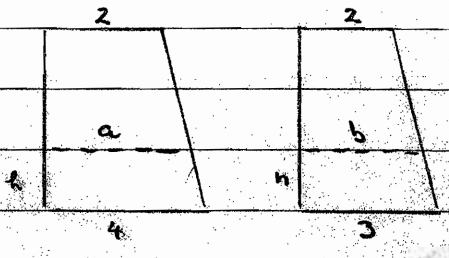
$V = 8 \int_1^0 u^4 \cdot -2(1-u) \, du$ 1

$= 16 \int_0^1 u^4 - u^5 \, du$

$= 16 \left[\frac{u^5}{5} - \frac{u^6}{6} \right]_0^1$ 1

$= 16 \times \frac{1}{30} = \frac{8}{15}\ u^3$ 1

c) i)



$a = ml + k$ $\quad$ $b = ml + l$

$h = 0 \quad a = 4$ $\quad$ $l = 0 \quad b = 3$

$\therefore k = 4$ $\quad$ $\therefore l = 3$

when $h = 20 \quad a = 2$ $\quad$ when $h = 20, \ b = 2$

$2 = 20m + 4$ $\quad$ $2 = 20m + 3$

$m = -\frac{1}{10}$ $\quad$ $m = -\frac{1}{20}$

$\therefore a = 4 - \frac{l}{10}$ 1 $\quad$ $\therefore b = 3 - \frac{l}{20}$ 1

Area $= \pi\left(4 - \frac{l}{10}\right)\left(3 - \frac{l}{20}\right)$ 1

Volume of slice $\pi\left(12 - \frac{l}{2} + \frac{l^2}{200}\right)\delta h$ 1

ii) $V = \pi \int_0^{20} 12 - \frac{l}{2} + \frac{l^2}{200} \, dh$

$= \pi \left[12l - \frac{l^2}{4} + \frac{l^3}{600} \right]_0^{20}$ 1

$= \pi \left[240 - 100 + \frac{40}{3} \right]$

$= \frac{460\pi}{3}\ u^3$ 1

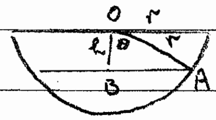
Q7) a) i) Consider a cross section radius r, height h

by Pythag third side $= \sqrt{r^2 - h^2}$

Surface Area $= \ell \times 2\sqrt{r^2 - h^2}$

$= 2\ell\sqrt{r^2-h^2}$ 1

ii)



Let $\theta = \hat{AOB}$

$\therefore$ AOB $= \cos\theta$

$\theta = \cos^{-1}\frac{h}{r}$ 1

Area of shaded segment

$= \frac{r^2}{2}(2\theta - \sin 2\theta)$

$= r^2(\theta - \sin\theta\cos\theta)$

$= r^2\left(\cos^{-1}\frac{h}{r} - \frac{\sqrt{r^2-h^2}}{r}\cdot\frac{h}{r}\right)$

Volume $= \ell\left[r^2\cos^{-1}\frac{h}{r} - h\sqrt{r^2-h^2}\right]$ 1

iii) $\frac{dV}{dt} = \ell\left[r^2 \times \frac{-\frac{1}{r}}{\sqrt{1-\frac{h^2}{r^2}}} - \sqrt{r^2-h^2} + \frac{h^2}{\sqrt{r^2-h^2}}\right]\frac{dh}{dt}$

$= \ell\left[\frac{-r^2 - (r^2-h^2) + h^2}{\sqrt{r^2-h^2}}\right]\frac{dh}{dt}$ 1

$= \frac{-2\ell(r^2-h^2)}{(r^2-h^2)^{1/2}}\frac{dh}{dt}$

$= -2\ell\sqrt{r^2-h^2}\,\frac{dh}{dt}$ 1

$= -A\frac{dh}{dt}$ 1

iv) $-\frac{dV}{dt} \propto A \quad \therefore -\frac{dV}{dt} = kA$

$\therefore kA = A\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = k$

$\therefore h$ is incr at a const rate. 1

b) i) $240t\cos\theta = 3600$

$\cos\theta\, t = 15$

$t = 15\sec\theta$ 1

$2000 + 240t\sin\theta - \frac{gt^2}{2} = 3200 - \frac{gt^2}{2}$

$240t\sin\theta = 1200$

$t\sin\theta = 5$

$\therefore \tan\theta = \frac{1}{3}$ 1

$\theta = \tan^{-1}\frac{1}{3} \doteq 18°26$

at time $t = 15\sec\theta$

$= 5\sqrt{10}$ s 1

ii) when $t = 5\sqrt{10}$ & $g = 10$

$H = 3200 - \frac{10.250}{2}$

$= 1950$ m 1

c) $1^3 + 2^3 + \ldots + n^3 \le n^2(n+1)^2$

Step 1. Prove true for $n = 1$

L.H.S. $= 1$ R.H.S. $(1 \cdot 4) = 4$

L.H.S < R.HS $\therefore$ true for $n=1$ 1

Step 2. Assume result true for $n = k$

ie $1^3 + 2^3 + \ldots + k^3 \le k^2(k+1)^2$

Prove true for $n = k+1$

$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = (k+1)^2(k+2)^2$ 1

L.H.S. $< k^2(k+1)^2 + (k+1)^3$

$= (k+1)^2(k^2+k+1)$

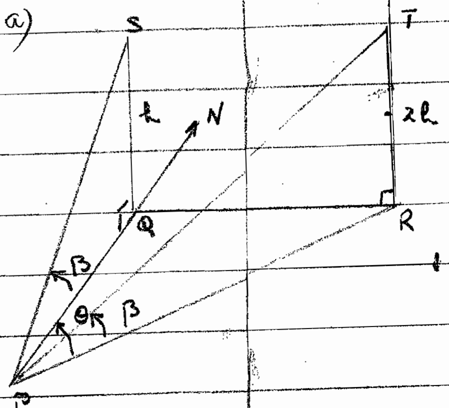
R.H.S. $= (k+1)^2(k^2+4k+4)$ 1

$\therefore$ L.H.S. < R.H.S.

Step 3. If the result is true for $n=1$ and it is true for $n=k+1$, then it is true for $n=2$ and so on using math^l indu

(nothing for concl.)

Q8) a)



$\Delta PSQ \parallel\!\mid \Delta PTR$ (equiangular)

$$\frac{PR}{PQ} = \frac{RT}{QS} = 2$$

$PQ = h \cot\beta$ 1

$PR = 2h \cot\beta$

$QR = SW = h\cot\alpha$ 1

By cosine rule

$$\cos\theta = \frac{PQ^2 + PR^2 - QR^2}{2\,PQ.PR}$$ 1

$$\therefore \cos\theta = \frac{h^2\cot^2\beta + 4h^2\cot^2\beta - h^2\cot^2\alpha}{2h\cot\beta\, 2h\cot\beta}$$

$$= \frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}$$ 1

Q8) b) i) Let $T\hat{R}P = \alpha$

$P\hat{S}Q = T\hat{R}P = \alpha$

opp. angle of a cyclic quad = opp. int. angle

$\therefore T\hat{R}P = T\hat{P}Q$ 1

In ΔTRP & ΔTQP

$T\hat{R}P = T\hat{P}Q$ (shown above) 1

$P\hat{T}R = Q\hat{T}P$ (common)

$\therefore \Delta TPR \parallel\!\mid \Delta TQP$ 1

ii) Join PS

SPYX is a cyclic quad with SY as diam. 1

$S\hat{Y}P = S\hat{X}P = 90°$ (angle in a semicircle) 1

iii) $P\hat{Y}X + P\hat{S}Q = 180$ (opp angles of a cyclic quad) 1

$\therefore P\hat{Y}X = 180 - \alpha$ 1

$P\hat{R}Q + T\hat{R}P = 180$ (st. angle)

$\therefore P\hat{R}Q = 180 - \alpha$ 1

$\therefore P\hat{R}Q = P\hat{Y}X$

(c) (i) $xy = c^2$ and $P\left(ct, \frac{c}{t}\right)$

$$y = c^2x^{-1}$$

$$\frac{dy}{dx} = -c^2x^{-2} \qquad \#$$

$$= -\frac{c^2}{x^2}$$

At P, gradient of tangent, m_1:

$$m_1 = -\frac{c^2}{c^2t^2} \Rightarrow m_1 = -\frac{1}{t^2}$$

The gradient of the normal at P is t^2.
Hence the equation of the normal is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{c}{t} = t^2(x - ct)$$

$$y = t^2x + \frac{c}{t} - ct^3 \qquad \text{... (1)}$$

Note: parametric differentiation is an alternative here.

(ii) Put $y = x$ ... (2)

Solving (1) and (2)

$$x = t^2x + \frac{c}{t} - ct^3$$

$$t^2x - x = \frac{ct^4 - c}{t}$$

$$x(t^2 - 1) = \frac{c(t^2 - 1)(t^2 + 1)}{t}$$

$$x = \frac{c}{t}(t^2 + 1) \text{ provided } t \neq \pm 1$$

Hence N is $\left(\frac{c}{t}(t^2 + 1), \frac{c}{t}(t^2 + 1)\right)$

(iii) Show that $OP = NP$ for triangle OPN to be isosceles.

Using $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$$OP^2 = (ct)^2 + \left(\frac{c}{t}\right)^2$$

$$OP^2 = c^2t^2 + \frac{c^2}{t^2}$$

$$OP^2 = c^2\left(t^2 + \frac{1}{t^2}\right)$$

$$OP^2 = c^2\left(\frac{t^4 + 1}{t^2}\right)$$

$$OP = \frac{c}{t}\sqrt{t^4 + 1}$$

$$NP^2 = \left[\frac{c}{t}(t^2+1) - ct\right]^2 + \left[\frac{c}{t}(t^2+1) - \frac{c}{t}\right]^2$$

$$NP^2 = c^2\left(\frac{t^2+1}{t} - t\right)^2 + \frac{c^2}{t^2}(t^2+1-1)^2$$

$$NP^2 = \frac{c^2}{t^2} + \frac{c^2}{t^2}(t^4)$$

$$NP = \frac{c}{t}\sqrt{1 + t^4}$$

Hence $NP = OP$ and so triangle OPN is isosceles.